Models of Set Theory II - Winter 2017/2018

Prof. Dr. Peter Koepke, Ana Njegomir Problem sheet 9

Problem 1 (5 points)**.** Let $\mathcal{F} \subseteq [\omega]^\omega$ be a family which satisfies the sfip and consider the forcing notion $\mathbb{P}_\mathcal{F}$ whose conditions are pairs $p = \langle s_p, E_p \rangle$ such that s_p is a finite subset of ω and E_p is a finite subset of $\mathcal{F}$, ordered by

$$p \leq q \iff s_p \supseteq s_q \wedge E_p \supseteq E_q \wedge s_p \setminus s_q \subseteq \bigcap E_q.$$

If G is M-generic for $\mathbb{P}_\mathcal{F}$ then prove that in $M[G]$, $\mathcal{F}$ has a pseudo-intersection.

Problem 2 (4 points)**.** Show that MA implies that $\mathfrak{p} = 2^{\aleph_0}$.

Problem 3 (3 points)**.** Prove that $\mathfrak{b} \leq \mathrm{non}(\mathcal{M})$.

Problem 4 (2 points)**.** Find a forcing notion $\mathbb{P}$ which decides the Continuum Hypothesis in the following way: There are $\mathbb{P}$-generic filters G and H such that $M[G] \vDash \mathrm{CH}$ and $M[H] \vDash \neg\mathrm{CH}$.

Definition. Suppose that S is an uncountable set and $\kappa > \omega$ is a cardinal. Suppose that $A \subseteq [S]^{<\kappa} = \{X \subseteq S \mid |X| < \kappa\}$.

(1) A is *unbounded* if for all $x \in [S]^{<\kappa}$ there is $y \in A$ with $x \subseteq y$.
(2) A is *closed* if for all $\subseteq$-*chains* $\langle x_\alpha \mid \alpha < \gamma \rangle$ of sets in A, i.e. $x_\alpha \subseteq x_\beta$ for $\alpha < \beta$, with $\gamma < \kappa$, $\bigcup_{\alpha<\gamma} x_\alpha \in A$.
(3) A is *stationary* if $A \cap C \neq \emptyset$ for every *club* (closed unbounded) $C \subseteq [S]^{<\kappa}$.
(4) A is *directed* if for all $x, y \in A$ there is $z \in A$ such that $x \cup y \subseteq z$.

Problem 5 (6 points)**.** If $f : [S]^{<\omega} \to [S]^{<\kappa}$ then we define

$$C_f = \{x \in [S]^{<\kappa} \mid \forall e \in [x]^{<\omega} (f(e) \subseteq x)\}$$

the set of *closure points of f*.

(a) Suppose that $C \subseteq [S]^{<\kappa}$ is closed and $A \subseteq C$ is directed with $|A| < \kappa$. Show that $\bigcup A \in C$.
(b) Show that for every club subset of $[S]^{<\kappa}$ there is $f : [S]^{<\omega} \to [S]^{<\kappa}$ such that $C_f \subseteq C$.
(c) Show that for every function $f : [S]^{<\omega} \to [S]^{<\kappa}$, C_f is club in $[S]^{<\kappa}$.

Please hand in your solutions on Monday, December 11 before the lecture.